Exam Probability and Measure (WBMA024-05) Monday June 19 2023, 15.00-17.00

This exam consists of 3 exercises all with subquestions. You can get 90 points in total and the grade for this written exam is obtained through

$$1 + \frac{\text{obtained points}}{10}.$$

The grade for this written exam contributes 60% to the final grade of the course.

Write your name and student number on every page you hand in and number the pages.

Throughout the exam $\mathbb{N} = \{1, 2, \cdots\}$ are the natural numbers. $\mathcal{B}$ is used for the Borel σ-algebra on $\mathbb{R}$.

Partial Answers may be worth points!!

Exercises with a $*$ are expected to be the hardest ones of this exam.

1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, where the σ-algebra $\mathcal{A}$ is generated by the finite partition $\mathcal{P} = \{A_1, A_2, \cdots, A_r\}$ of Ω. Let $f : \Omega \to \mathbb{R}$ be an $(\mathcal{A}, \mathcal{B})$-measurable function. Show that there exist $\alpha_1, \cdots, \alpha_r \in \mathbb{R}$, such that f may be written as

$$f = \sum_{i=1}^{r} \alpha_i \mathbb{1}_{A_i}.$$

That is, show that f is constant on the separate elements of $\mathcal{P}$. (15pt)

2. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, with $\mu(\Omega) = \infty$ and let $f : \Omega \to \mathbb{R}$ be an $(\mathcal{A}, \mathcal{B})$ measurable function. Assume $f \in \mathcal{L}^p(\Omega)$ for all $p \in [1, \infty]$ and that $\|f\|_\infty > 0$.

a) Provide the definitions of $\|f\|_\infty$ and $\|f\|_p$. (10pt)

b) Show that $\|f\|_\infty \leq \liminf_{p\to\infty} \|f\|_p$. (10pt)

Hint: Take the integral of some useful function over $\Gamma = \{\omega \in \Omega : |f(\omega)| > M\}$, where $M \in (0, \|f\|_\infty)$. Relate this integral to $\|f\|_\infty$ and $\|f\|_p$ for given finite p.

c) Formulate Hölder's inequality. (5pt)

d*) Show that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. (10pt)

Hint: Show that $\|f\|_p \leq (\|f\|_1)^{1/p} (\|f\|_\infty)^{1-1/p}$ and deduce that $\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty$. Combine this with part b).

3. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$, such that

$$\mathbb{P}\left(\bigcap_{i\in\mathcal{I}} A_i\right) = \prod_{i\in\mathcal{I}} \mathbb{P}(A_i) \qquad \text{for all finite } \mathcal{I} \subset \mathbb{N}.$$

That is, the A_i's are mutually independent. Let

$$\begin{aligned} A &= \{\omega \in \Omega : \omega \in A_i \text{ for infinitely many values of } i\} \\ &= \{\omega \in \Omega : \text{for all } n \in \mathbb{N} \text{ there is an } i \geq n \text{ such that } \omega \in A_i\}. \end{aligned}$$

a) Show that $A \in \mathcal{A}$. (10pt)

b) Show that if $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \infty$, then $\mathbb{P}(A) = 1$. (10pt)

Hint: You may use $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$.

Let C be a circle with circumference 1 (i.e. with radius $1/(2\pi)$). One by one randomly chosen closed arcs (on C) denoted by I_i ($i \in \mathbb{N}$) of respective lengths ℓ_i are colored red. Assume further that $1 > \ell_1 \geq \ell_2 \geq \cdots$. Denote the midpoint of I_i by x_i. The x_i are chosen independently and uniformly on C. Let S be the part of the circle that is colored red. That is,

$$S = \cup_{i=1}^{\infty} I_i.$$

Note that the uniformity of the x_i's implies that for every $c \in C$ and $i \in \mathbb{N}$ we have $\mathbb{P}(c \in I_i) = \ell_i$. Define

$$k_n = 2^{n+1} \times n! \quad \text{and} \quad K_n = \sum_{i=1}^{n} k_i \qquad \text{both for } n \in \mathbb{N}.$$

For $K_{n-1} < i \leq K_n$ let $\ell_i = 1/(2k_n)$. So, there are k_n arcs of length $1/(2k_n)$.

c) Show that $\mathbb{P}(c \in S) = 1$ for all $c \in C$, and that C will eventually be red almost everywhere (with respect to Lebesque measure on C). (10pt)

d*) Show that $\mathbb{P}(C = S) < 1$. (10pt)

Hint: You may use the following approach:
Show that with positive probability, there is an infinite sequence of non-empty arcs $a_1 \supset a_2 \supset \cdots$ of respective lengths $1/(2k_1), 1/(2k_2), \cdots$, such that $a_n \cap (\cup_{i=1}^{K_n} I_i) = \emptyset$. You can do this by first showing that a_1 exists with strictly postive probability and then condition on that a_n exists. Then split a_n in $k_{n+1}/(2k_n)$ disjoint arcs of length $1/k_{n+1}$, and show that with "desirable" probability at least one of those arcs does not contain any of the x_i for $i \leq K_{n+1}$.
You may use without further proof that there exists $z \in (0, 1)$, such that out of any subset of j disjoint arcs of length $1/k_{n+1}$ the probability that at least one of those arcs does not contain any of the x_i with $K_n < i \leq K_{n+1}$ is larger than $1 - z^{-j}$ for all $n \in \mathbb{N}$.